JOURNAL OF APPROXIMATION THEORY 35, 94–97 (1982)

Note

A Generalization of an Inequality of V. Markov to Multivariate Polynomials

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Communicated by T. J. Rivlin

Received March 23, 1981

If P_n^r is a polynomial of total degree n ($\geqslant 2$) in r ($\geqslant 1$) variables then the sum of those coefficients whose degree is precisely $n-1$ cannot exceed $2^{n-2}\|P_n^r\|$ in asolute value (the uniform norm is taken on the r-dimensional unit cube). This generalizes a well-known inequality for univariate polynomials which is due to V. Markov.

1. INTRODUCTION

P. Chebyshev's [1] celebrated theorem on polynomials with fixed leading coefficients which deviate least from zero on $I=[-1,1]$ has the reinterpretation

$$|a_n| \leqslant 2^{n-1}\|F_n\|, \tag{1}$$

where $P_n(x)=\sum_{j=0}^n a_j x^j$ is a univariate polynomial of degree not higher than $n \geqslant 1$ and $\|\cdot\|$ denotes the uniform norm on I. Markov [2] obtained estimates for the other coefficients of P_n as well (see also [3, p. 56] or [4, p. 167]). In particular,

$$|a_{n-1}| \leqslant 2^{n-2}\|P_n\|, \qquad n \geqslant 2. \tag{2}$$

There is equality in (1) or (2) if $P_n=T_n$ or T_{n-1}, respectively, $T_n(x)=\cos(n \arccos x)$ being the nth Chebyshev polynomial of the first kind.

In this note we consider extensions of (1) and (2) to multivariate polynomials.

Denote by $P_n^r(\mathbf{x})=\sum_{\mathbf{m}'\leqslant n} b_{\mathbf{m}}\mathbf{x}^{\mathbf{m}}$ a polynomial in r variables of total degree not exceeding n. Here, $\mathbf{x}=(x_1,..., x_r)\in \mathbb{R}^r$, $\mathbf{m}=(m_1,..., m_r)\in \mathbb{N}_0^r$, $\mathbf{x}^{\mathbf{m}}=$

0021-9045/82/050094–04$02.00/0

$x_1^{m_1} \cdots x_r^{m_r}$, $b_{\mathbf{m}} \in \mathbb{R}$, and $\mathbf{m}' = m_1 + \cdots + m_r$. We put $\|P_n^r\| = \max |P_n^r(\mathbf{x})|$, $\mathbf{x} \in I^r$.

2. Result

Our first aim is to call attention to a generalization of (1) which is contained in [5] (see also [6, p. 234]).

THEOREM 1 (C. Visser)

$$\left| \sum_{\mathbf{m}'=n} b_{\mathbf{m}} \right| \leqslant 2^{n-1} \|P_n^r\|. \tag{1'}$$

We add here the corresponding generalization of (2).

THEOREM 2.

$$\left| \sum_{\mathbf{m}'=n-1} b_{\mathbf{m}} \right| \leqslant 2^{n-2} \|P_n^r\|, \qquad n \geqslant 2. \tag{2'}$$

There is equality in (1′) or (2′) if $P_n^r(\mathbf{x}) = \sum_{j=1}^r T_n(x_j)$ or $\sum_{j=1}^r T_{n-1}(x_j)$, respectively, equality in (1′) or (2′) also holds for T_n or T_{n-1}, respectively, when regarded as polynomials in several variables.

We note in passing that estimates for each single leading coefficient $b_{\mathbf{m}}$ ($\mathbf{m}' = n$) were given by M. Reimer [7].

3. Proof

We shall prove Theorem 2 using a modification of the argument in [5] which in turn is an extension of an argument already used by H. Liebmann [8] for a proof of (1); see also L. Fejér [9, p. 82]. The following discrete orthogonality relationships of the exponential function are well known:

$$\begin{aligned} \sum_{p=0}^{N-1} e^{2\pi i(v-u)p/N} &= N, \qquad \text{if} \quad u \equiv v \bmod N,\ N \in \mathbb{N}, \\ &= 0, \qquad \text{otherwise.} \end{aligned} \tag{3}$$

Choosing $N = 2(n-1)$, $u = n-1$ and $v = \mathbf{k}'$ for some $\mathbf{k} \in \mathbb{Z}^r$ yields

$$\begin{aligned} \sum_{p=0}^{2n-3} (-1)^p e^{i\langle \mathbf{k}, \mathbf{s} \rangle} &= 2(n-1), \qquad \text{if} \quad n-1 \equiv \mathbf{k}' \bmod 2(n-1), \\ &= 0, \qquad \text{otherwise,} \end{aligned} \tag{4}$$

with $\mathbf{s} = \mathbf{s}(p) = (p\pi/(n-1), \ldots, p\pi/(n-1)) \in \mathbb{R}^r$ and $\langle , \rangle$ denoting the usual inner product in $\mathbb{R}^r$.

We put $P_n^r(\mathbf{x}) = P_n^r(\cos t_1, \ldots, \cos t_r)$. Then P_n^r transforms into a trigonometric sum F which can be written as

$$F(\mathbf{t}) = \sum_{|\mathbf{k}'| \leqslant n} c_\mathbf{k} e^{i\langle \mathbf{k}, \mathbf{t} \rangle}, \qquad c_\mathbf{k} = c_{-\mathbf{k}}. \tag{5}$$

The coefficients of P_n^r and F satisfy, in particular, the identity

$$b_\mathbf{m}/2^{n-1} = c_\mathbf{k}, \qquad \text{if} \quad \mathbf{m}' = \mathbf{k}' = n-1. \tag{6}$$

We deduce that in view of (4), (5), and (6),

$$\begin{aligned}
\sum_{p=0}^{2n-3} (-1)^p F(\mathbf{s}) &= \sum_{p=0}^{2n-3} (-1)^p \left(\sum_{|\mathbf{k}'| \leqslant n} c_\mathbf{k} e^{i\langle \mathbf{k}, \mathbf{s} \rangle} \right) \\
&= \sum_{|\mathbf{k}'| \leqslant n} \left(c_\mathbf{k} \sum_{p=0}^{2n-3} (-1)^p e^{i\langle \mathbf{k}, \mathbf{s} \rangle} \right) \\
&= 2(n-1) \left(\sum_{\mathbf{k}' = -(n-1)} c_\mathbf{k} + \sum_{\mathbf{k}' = n-1} c_\mathbf{k} \right) \\
&= (2(n-1)/2^{n-2}) \left(\sum_{\mathbf{m}' = n-1} b_\mathbf{m} \right).
\end{aligned} \tag{7}$$

It follows that

$$\begin{aligned}
(2(n-1)/2^{n-2}) \left| \sum_{\mathbf{m}' = n-1} b_\mathbf{m} \right| &\leqslant \sum_{p=0}^{2n-3} |F(\mathbf{s})| \\
&\leqslant 2(n-1) \max_{\mathbf{t}} |F(\mathbf{t})| = 2(n-1) \, \|P_n^r\|.
\end{aligned}$$

Note that the above argument does not provide estimates for

$$\left| \sum_{\mathbf{m}' = n-q} b_\mathbf{m} \right|, \tag{8}$$

$q \geqslant 2$, since now a one-to-one correspondence between the coefficients of P_n^r and F as that in (6) fails to hold. Theorem 2 is excerpted from the author's forthcoming doctoral dissertation at Universität Dortmund.

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Printed in Belgium